Math 259A Lecture 14 Notes

Daniel Raban

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1 Geometry of Projections

1.1 Geometry of projections in a von Neumann algebra

Let $M \subseteq \mathcal{B}(H)$ be a von Neumann algebra, and let $P(M)$ be the projections in M.

Definition 1.1. Projections $e, f \in P(M)$ are **equivalent** ($e \sim f$) if there exists a partial isometry $v \in M$ such that $vv^* = e$ and $v^*v = f$ (i.e. $\ell(v) = e$, $r(v) = f$). We say that e is **dominated** by f ($e \prec f$) if there exists $f_1 \leq f$ such that $e \sim f_1$ (i.e. there is a partial isometry $v \in M$ such that $vv^* = e$ and $v^*v \leq f$.

Proposition 1.1. *For all $x \in M$, $\ell(x) \sim r(x)$.*

Proof. This follows from the polar decomposition of x: $x = v|x|$. Then $s(|x|) = r(x)$, and $|x| \in s(|x|)Ms(|x|)$. □

Theorem 1.1 (Paralellogram law)**.** *If $e, f \in P(M)$, then $(e \vee f - f) \sim (e - e \wedge f)$.*

Proof. The left hand side is $\ell(e(1-f))$, and the right hand side is $r(e(1-f))$. □

Definition 1.2. The **center** of M is $Z(M) = M' \cap M$.

Definition 1.3. Let $x \in M$. The **central support** of x is the smallest projection z in $Z(M)$ such that $zx = x = xz$. We denote this by $z(x)$.

By taking $\bigwedge_{z_i x = x} z_i$, this exists.

Proposition 1.2. $z(x) = [MxH]$.

Proof. Call the right hand side the projection p. Since $zx = x$, $z \geq p$: $z = uzu^*$ where u is unitary, so $z \geq u\ell(x)u^*$ for all unitary u. So $z \geq \bigvee_u u\ell(x)u^* = [MxH]$ because $\operatorname{span}(U(M)) = M$.

But $px = x$, and $p \in Z(M)$ because $M'MxH = MxH$ and $MMxH = MxH$. By the definition of z, $p \geq z(x)$. □

Theorem 1.2. *Let $e, f \in P(M)$. The following are equivalent:*

1. $eMf \neq 0$.
2. *there exist a nonzero* $e_1 \leq e$ *and a nonzero* $f_1 \leq f$ *with* $e_1 \sim f_1$.
3. $z(e)z(f) \neq 0$.

Theorem 1.3 (Comparison theorem)**.** *Let $e, f \in P(M)$. There exists a projection in $Z(M)$ such that $ep \prec fp$ or $e(1-p) \succ f(1-p)$.*

Proof. Exercise.[1] ☐

Corollary 1.1. *If $Z(M) = \mathbb{C}$, then $e \prec f$ or $e \succ f$.*

Theorem 1.4 (Schröder-Bernstein type theorem)**.** *If $e \succ f$ and $f \succ e$, then $e \sim f$.*

Proof. Exercise.[2] ☐

1.2 Vold decomposition

Example 1.1. The left shift on $\ell^2(\mathbb{N})$ is an isometry.

Theorem 1.5 (Vold's decomposition theorem)**.** *If $v \in M$ is an isometry (i.e. $v^*v = 1$), then $v = u \oplus v_0$, where there is a projection p with $u^*u = uu^* = p$, $v_0^*v_0 = 1-p$, $v_0v_0^* \leq 1-p$. (So $v_0^n(v_0^*)^n$ is a decreasing sequence of projections decreasing to 0. This decomposition is unique.*

Remark 1.1. If p is as above, $v^n(v^*)^n \searrow p$. In particular, if $p_0 = (1-p) - v_0v_0^*$, then all $v^n p_0 (v^n)^*$ are mutually orthogonal.

1.3 Factors and finite projections

Definition 1.4. M is a **factor** if $Z(M) = \mathbb{C}$.

Definition 1.5. $e \in P(M)$ is **abelian** if eMe is abelian.

Example 1.2. $e \in B(H)$ is abelian if and only if e is a 1-dimensional projection.

Definition 1.6. $e \in P(M)$ is a **finite projection** if whenever $f \leq e$ and $f \sim e$, $f = e$.

This is like saying that a set E is finite if the only subset of E that it is in bijection with is E itself.

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[1] :(
[2] :(

Remark 1.2. This is equivalent to the following: for any partial isometry $v \in eMe$ with $v^*v = e$, we have $vv^* = e$; i.e. any isometry on eMe is a unitary in eMe.

Definition 1.7. $e \in P(M)$ is **properly infinite** if e has no direct summands in M that are finite, i.e. if $p \in P(M) \cap Z(M)$ with pe finite, then $pe = 0$.

Example 1.3. Consider the von Neumann algebra $\mathbb{C}1 \oplus \mathcal{B}(\ell^2(\mathbb{N}))$. Then $e = 1$ is not a finite projection, but it is not properly infinite. If p is the projection onto the $\mathcal{B}(\ell^2(\mathbb{N}))$ part, then p is properly infinite.

Definition 1.8. A von Neumann algebra M is **finite** if 1 is a finite projection (i.e. any isometry is necessarily a unitary). M is **semifinite** if $1_M = \bigvee_i e_i$ with e_i finite.

Example 1.4. $L^\infty(X)$ is finite (and so is any abelian von Neumann algebra).

Example 1.5. $M_n(\mathbb{C}) = \mathcal{B}(\ell_n^2)$ is finite.

Definition 1.9. A von Neumann algebra M is **type I** if $1_M = \bigvee_i e_i$ with e_i abelian.

Example 1.6. $\mathcal{B}(\ell^2(\mathbb{N}))$ is type I.

Next time, we willl discuss type II and type II von Neumann algebras.